# Verificación de Programas Concurrentes Usando Lógica
# *Logical Foundations of Concurrent Computation*

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ECI 2026
(Version of July 27, 2026)

# Preambulo

# Outline

# Verificación de Programas Concurrentes

- Muchos sistemas informáticos son **concurrentes**, y están compuestos de programas que se ejecutan simultáneamente.

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  - Sistemas distribuidos y paralelos
  - Arquitecturas basadas en microservicios
  - Go / Erlang / Rust / Akka / Elixir

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- Establecer la correctitud de estos programas implica garantizar que las interacciones entre ellos respetan los **protocolos establecidos**.
- Una tarea crucial, pero también compleja: necesitamos técnicas de verificación capaces de analizar **estructuras de comunicación** sofisticadas.

# Verificación de Programas Concurrentes **Usando Lógica**

**Este Curso**

- Técnicas fundamentales para la verificación para programas concurrentes.
- **Enfasis:** Conceptos de la **lógica** para el análisis y verificación de programas.

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## Ideas Principales

- Describir los protocolos de comunicación usando **sistemas de tipos**.
- Especificar los programas concurrentes con un **lenguaje formal de procesos**.
- Los tipos aseguran que los procesos **usan sus recursos correctamente**.

$\rightarrow$ Uso adecuado de recursos = ejecución correcta de protocolos.

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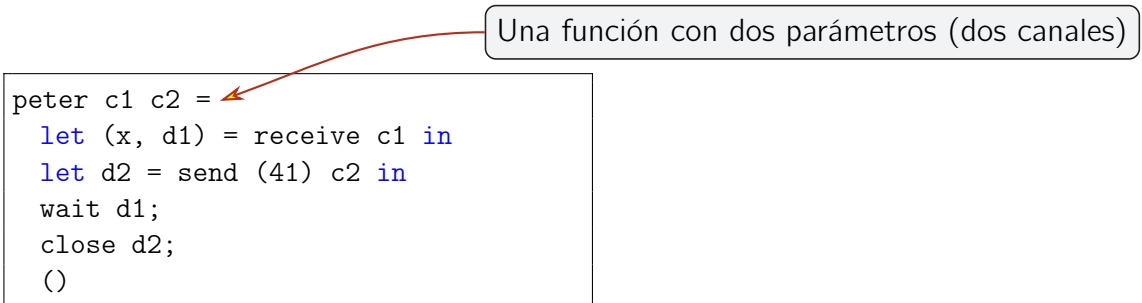
## Correspondencia de Curry-Howard

Demostrar un teorema y escribir un programa correcto son la **misma actividad** desde perspectivas distintas.

# Ejemplo: Un Programa con Paso de Mensajes

```
peter c1 c2 =
  let (x, d1) = receive c1 in
  let d2 = send (41) c2 in
  wait d1;
  close d2;
  ()
```

# Ejemplo: Un Programa con Paso de Mensajes



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  ()
```

Recibe un mensaje en c1, continúa en d1

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  let (x, d1) = receive c1 in
  let d2 = send (41) c2 in
  wait d1;
  close d2;
  ()
```

Envía ‘41’ en c2, continúa en d2

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peter c1 c2 =
  let (x, d1) = receive c1 in
  let d2 = send (41) c2 in
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  ()
```

Espera confirmación para cerrar canal d1

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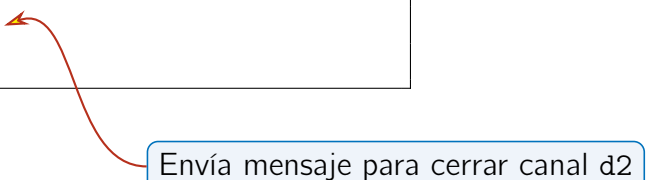

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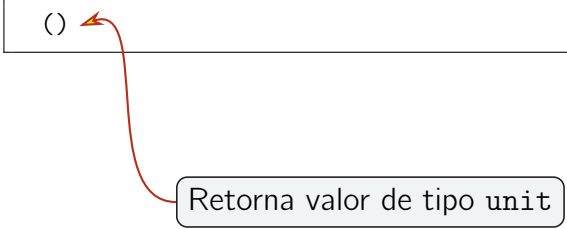

# Ejemplo: Dos Programas con Paso de Mensajes

```
peter c1 c2 =
  let (x, c1) = receive c1 in
  let c2 = send (41) c2 in
  wait c1;
  close c2;
  ()
```

```
miles c1 c2 =
  let (x, c2) = receive c2 in
  let c1 = send (42) c1 in
  wait c2;
  close c1;
  ()
```

# Ejemplo: Dos Programas con Paso de Mensajes

```
peter c1 c2 =
  let (x, c1) = receive c1 in
  let c2 = send (41) c2 in
  wait c1;
  close c2;
  ()
```

El programa respeta su protocolo ✔

```
miles c1 c2 =
  let (x, c2) = receive c2 in
  let c1 = send (42) c1 in
  wait c2;
  close c1;
  ()
```

# Ejemplo: Dos Programas con Paso de Mensajes

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peter c1 c2 =
  let (x, c1) = receive c1 in
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El programa respeta su protocolo ✔

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  ()
```

El programa también respeta su protocolo ✔

# Ejemplo: Interacción de Programas

```
peter c1 c2 =
  let (x, c1) = receive c1 in
  let c2 = send (41) c2 in
  wait c1;
  close c2;
  ()
```

```
miles c1 c2 =
  let (x, c2) = receive c2 in
  let c1 = send (42) c1 in
  wait c2;
  close c1;
  ()
```

```
main =
  let (c1, c1dual) = new @T () in
  let (c2, c2dual) = new @T () in
  fork (miles c1dual c2);
       peter c1 c2dual
```

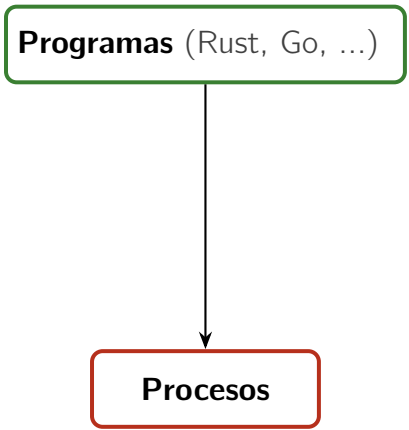
Es 'main' correcto?

# Verificación de Programas usando Sistemas de Tipos

**Programas** (Rust, Go, ...)

**Procesos**

# Verificación de Programas usando Sistemas de Tipos



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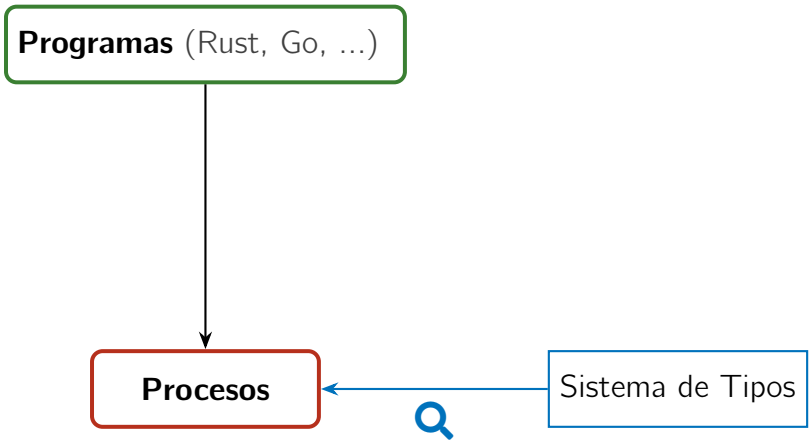

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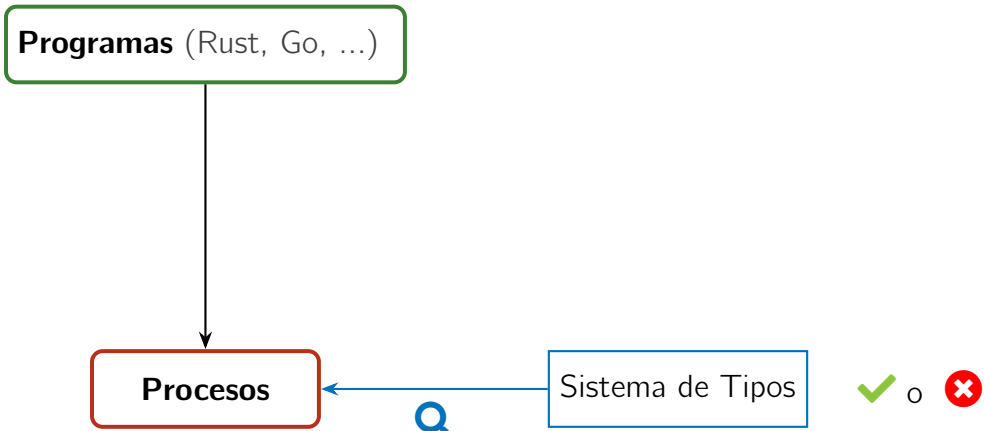

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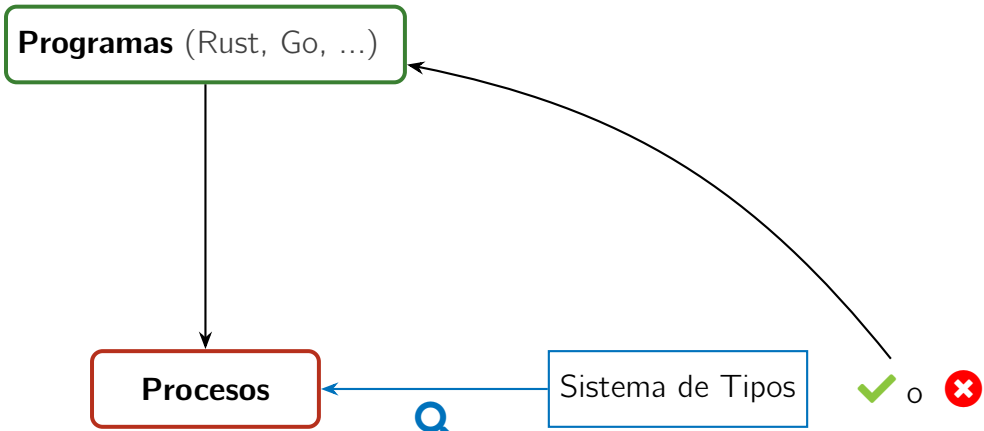

# Outline

# Un Lenguaje Formal para Concurrencia: Cálculos de Procesos

Asumiendo un conjunto de **nombres** $x, y, z, \ldots$ (usados para representar **canales**), definimos un lenguaje formal de **procesos** (denotados $P, Q, \ldots$):

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\begin{array}{rcll}
P, Q ::= & \overline{x}\langle y \rangle.P & \textbf{envia} \text{ mensaje } y \text{ en el canal } x, \text{ continúa con } P \\
\mid & x(y).P & \textbf{recibe} \text{ un mensaje } z \text{ en el canal } x, \text{ continúa con } P\{^z/_y\}
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\mid & \overline{x}\langle\rangle.P & \textbf{envía} \text{ una señal para cerrar el canal } x, \text{ continúa como } P \\
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| | | |
|---|---|---|
| $P, Q ::=$ | $\overline{x}\langle y\rangle.P$ | **envia** mensaje $y$ en el canal $x$, continúa con $P$ |
| $\mid$ | $x(y).P$ | **recibe** un mensaje $z$ en el canal $x$, continúa con $P\{^z/_y\}$ |
| $\mid$ | $\overline{x}\langle\rangle.P$ | **envía** una señal para cerrar el canal $x$, continúa como $P$ |
| $\mid$ | $x().P$ | **recibe** señal para cerrar el canal $x$, continúa como $P$ |
| $\mid$ | $P \mid Q$ | **concurrencia**: $P$ y $Q$ se ejecutan en paralelo |
| $\mid$ | $(\boldsymbol{\nu} x)P$ | **restricción**: el canal $x$ es local (privado) en $P$ |
| $\mid$ | $0$ | **cero**: el proceso que no hace nada |

# Un Lenguaje Formal: Cálculos de Procesos

Sintaxis de procesos:

$$P, Q ::= \overline{x}\langle y\rangle.P \mid x(y).P \mid \overline{x}\langle\rangle.P \mid x().P \mid P \mid Q \mid (\nu x)P \mid 0$$

Cómo expresar interacción?

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Cómo expresar interacción?
La **relación de reducción** formaliza la comunicación entre procesos concurrentes.
Dos reglas de la definición:

$$\overline{x}\langle z\rangle.P \mid x(y).Q \longrightarrow P \mid Q\{^z/_y\}$$

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**Intuición:**
Acciones que ocurren en paralelo (y en el mismo canal) pueden dar lugar a una sincronización. Las acciones deben ser complementarias (input / output).

# Ejercicio: Cuáles procesos pueden reducir (interactuar)?

1) $x(z).P \mid x(u).Q$

2) $\overline{x}\langle z\rangle.P \mid z(u).Q$

3) $\overline{x}\langle\rangle.x().P$

4) $\overline{x}\langle v_1\rangle.x(z).P \mid x(y).\overline{x}\langle v_2\rangle.Q$

5) $\overline{x}\langle z\rangle.0 \mid x(y).y(u).Q_1 \mid x(y).y().0$

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no hay reducción:

las acciones no son complementarias

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4) $\overline{x}\langle v_1\rangle.x(z).P \mid x(y).\overline{x}\langle v_2\rangle.Q$ — hay dos reducciones

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| 5) $\overline{x}\langle z\rangle.0 \mid x(y).y(u).Q_1 \mid x(y).y().0$ | hay una reducción:<br>el canal $z$ es un mensaje! |

# Ejercicio: Cuáles procesos pueden reducir (interactuar)?

Veamos (4) en detalle:

$$\begin{aligned} \overline{x}\langle v_1\rangle.x(z).P \mid x(y).\overline{x}\langle v_2\rangle.Q &\longrightarrow x(z).P \mid \overline{x}\langle v_2\rangle.Q\{^{v_1}/_{y}\} \\ &\longrightarrow P\{^{v_2}/_{z}\} \mid Q\{^{v_1}/_{y}\} \end{aligned}$$

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(5) es interesante: hay dos posibles reducciones, pero sólo una puede ocurrir. Por un lado tenemos:

$$\overline{x}\langle z\rangle.0 \mid x(y).y(u).Q_1 \mid x(y).y().0 \longrightarrow 0 \mid z(u).Q_1\{^{z}/_{y}\} \mid x(y).y().0$$

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pero también:

$$\overline{x}\langle z\rangle.0 \mid x(y).y(u).Q_1 \mid x(y).y().0 \longrightarrow 0 \mid x(y).y(u).Q_1 \mid z().0$$

# Un Lenguaje Formal: Cálculos de Procesos

Regresando al programa `main`, el protocolo de comunicación correspondiente se puede expresar de forma formal y compacta usando procesos:

$$(\boldsymbol{\nu} c)(\boldsymbol{\nu} d)(c(x).\overline{d}\langle 41\rangle.c().\overline{d}\langle\rangle.0$$
$$\mid$$
$$d(x).\overline{c}\langle 42\rangle.d().\overline{c}\langle\rangle.0)$$

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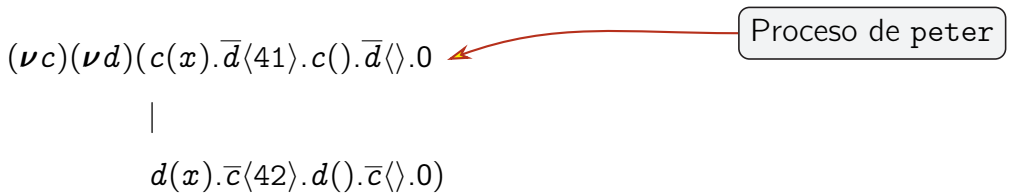

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$$\mid$$

$$d(x).\overline{c}\langle 42\rangle.d().\overline{c}\langle\rangle.0)$$

Proceso de `peter`

Proceso de `miles`

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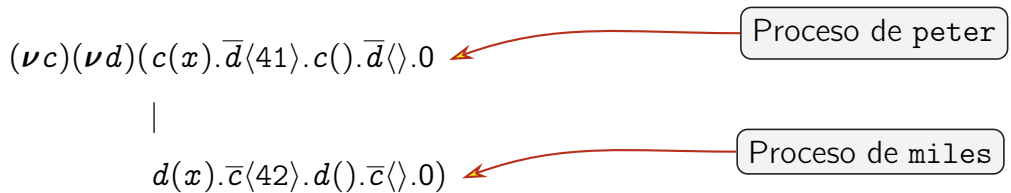


El análisis del sistema completo nos permite detectar una dependencia circular entre los dos sub-procesos: el programa no puede reducir y llega a un **deadlock**.

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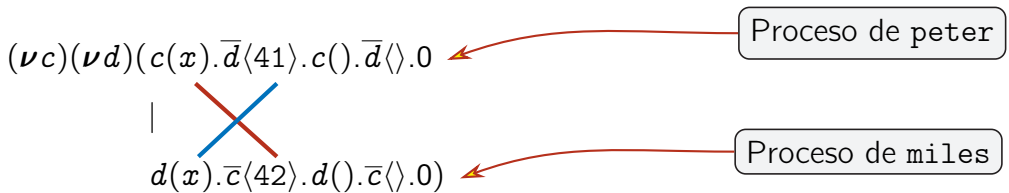


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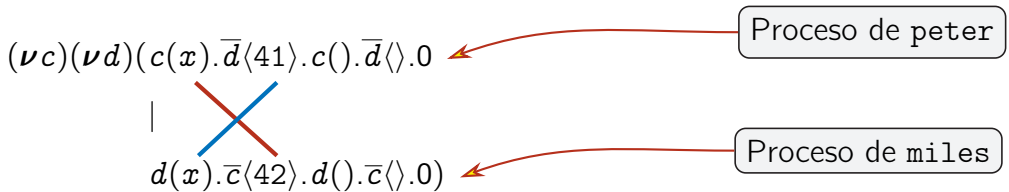


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**Este curso**

Presentamos el uso de lógica para la verificación estática de procesos concurrentes, asegurando que los protocolos se ejecutan correctamente, sin llegar a *deadlocks*.

# Outline

# Plan del Curso

1. **Session types**: Especificaciones precisas de protocolos para canales.

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6. Extensión de la correspondencia con recursos ilimitados: **clientes y servidores**.
7. La lógica de **implicaciones agrupadas** (*bunched implications*).

# Aspectos Prácticos

Cada clase estará dividida en tres partes:

- 1:30 - 2:15: Parte 1
- 2:15 - 2:20: Pausa
- 2:20 - 3:00: Parte 2
- 3:00 - 3:20: Coffee break
- 3:20 - 4:30: Parte 3

Material asociado al curso, actualizado después de cada clase:

`https://jperez.nl/teaching/eci26`

Preguntas, sugerencias, consultas:

`j.a.perez@rug.nl` (subject: [eci26])

# Session Types

# This Course: Session Types / Propositions as Sessions

An introduction to **session types** for concurrency, and to the logical foundations of message-passing computation.

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- A disciplined usage of sessions is key to ensure program correctness.
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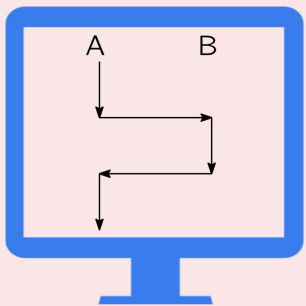
- **Linear Logic (LL)**: A resource-oriented view on propositions / protocols.
  Linear resources can be used exactly once.

In the rest of the lecture: the language of session types, and basics of linear logic.

# Outline

# When is a Program Correct?

## Sequential Programs




"Programs produce outputs that are consistent with their input"

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**An (imperfect) analogy**:

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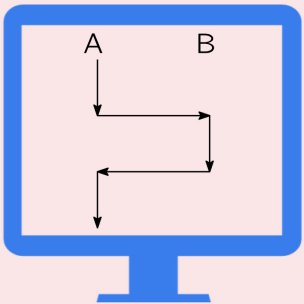


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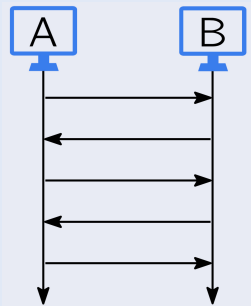

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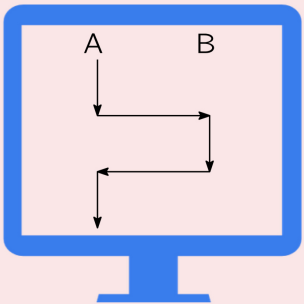


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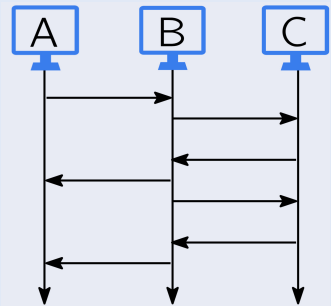

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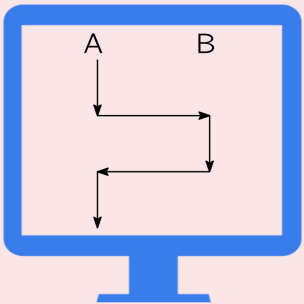


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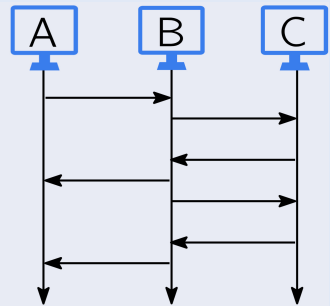


"Programs always respect their intended **protocols**"

# Example: A Two-Buyer Protocol


**Alice** and **Bob** cooperate in buying a book from **Seller**

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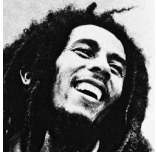


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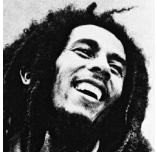

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**Note:** The structure of messaging matters!

# Example: A Two-Buyer Protocol

Correctness goals for the implementations of Alice, Bob, and Seller:

- **Fidelity** – they **follow the intended protocol**.
  - Alice never ask Bob twice within the same conversation
  - Alice doesn't continue the transaction if Bob can't contribute
  - Alice chooses among the options provided by Seller

# Example: A Two-Buyer Protocol

Correctness goals for the implementations of Alice, Bob, and Seller:

- **Fidelity** – they **follow the intended protocol**.
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  - Seller always returns an integer when Alice requests a quote

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Correctness goals for the implementations of Alice, Bob, and Seller:

- **Fidelity** – they **follow the intended protocol**.
- **Safety** – they don't feature **communication errors**.
- **Deadlock-Freedom** – they do not **"get stuck"** while running the protocol.
  - Alice eventually receives an answer from Bob on his contribution.

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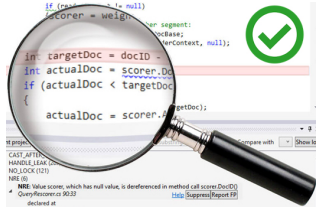
## Correctness

- A **non-trivial notion**, which follows from the interplay of these properties.
- **Hard to enforce** due to actions being "scattered around" in programs.

→ A session specifies a protocol's structure, enabling program verification. It stipulates **what** and **when** should be exchanged (along a channel)

# Type Systems

- Can detect bugs before programs are run
- Attached to many programming languages
- Implement a specific notion of correctness
  A program is either correct or incorrect



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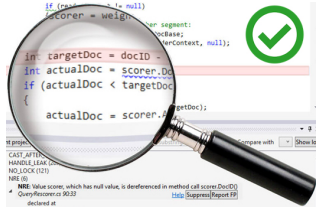


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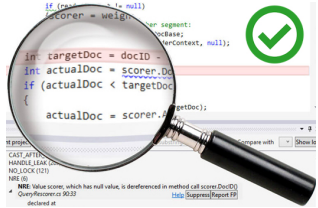


## Sequential Languages

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## Concurrent Languages

- **Behavioral** type systems classify protocols in a program
- Example: “first send username, then receive true/false, finally close”
- A typical bug: sending messages in the wrong order

# How to Design Type Systems? Logic to the Rescue!

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A landmark result in programming language theory:

- **Propositions as types** (Curry, 1935; Howard, 1969)

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| Propositions in Intuitionistic Logic | $\Longleftrightarrow$ | Types |
| Proofs in natural deduction | $\Longleftrightarrow$ | Sequential programs ($\lambda$-calculus) |
| Proof simplification | $\Longleftrightarrow$ | Program evaluation ($\beta$-reduction) |

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- What about concurrent programs? A **resource-oriented** view!

# How to Design Type Systems? Logic to the Rescue!

- **Propositions as sessions**

| | | |
|---|---|---|
| Propositions in **Linear Logic** (LL) | $\leftrightarrow$ | **Session types** |
| Proofs in LL | $\leftrightarrow$ | Interacting processes |
| Cut elimination in LL | $\leftrightarrow$ | process communication |

# Outline

# Protocols as Session Types

Session types specify protocols in terms of

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- choices: offers and selections
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Session protocols are attached to **interaction devices**:

- communication channels in programs (think Go and Rust)
- TCP-IP sockets
- ...

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A formal syntax for protocols:

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**Notice**:

- Sequential communication patterns (no built-in concurrency).
- $U$ stands for basic values (e.g. `int`) but also other sessions $S$.

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# The Two-Buyer Protocol, Revisited

Rakuten kobo

Recall the protocol between Alice, Bob, and Seller:

1. Alice sends a book title to Seller, who sends a quote back.
2. Alice checks whether Bob can contribute to buy the book.
3. Alice uses the answer from Bob (yes/no) to interact with Seller, either:
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# Session Types for The Two-Buyer Protocol

Two independent protocols, with Alice "leading" the interactions:

1. A session type for Seller (in its interaction with Alice):

$$S_{\mathsf{SA}} = ?\texttt{book};\ !\texttt{quote};\ \&\begin{cases} \text{buy}: & ?\texttt{paym};\ ?\texttt{address};\ !\texttt{ok};\ \text{end} \\ \text{cancel}: & ?\texttt{thanks};\ !\texttt{bye};\ \text{end} \end{cases}$$

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2. A session type for Alice (in its interaction with Bob):

$$S_{\mathsf{AB}} = \texttt{!cost}; \& \begin{cases} \text{share}: & \texttt{?address}; \texttt{!ok}; \text{end} \\ \text{close}: & \texttt{!bye}; \text{end} \end{cases}$$

# Session Types for The Two-Buyer Protocol

Implementations for Alice, Bob, and Seller should be **compatible**.

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**Duality** relates session types with opposite behaviors:

- The dual of sending is receiving (and vice versa)
- Branching is the dual of selection (and vice versa)

The dual of session type $S$ is written $\overline{S}$.

# Session Types for The Two-Buyer Protocol

**Example**:

- Recall that $S_{\mathsf{AB}}$ describes Alice's viewpoint in her interaction with Bob:

$$S_{\mathsf{AB}} = \; !\texttt{cost}; \& \begin{cases} \text{share}: & ?\texttt{address}; \; !\texttt{ok}; \; \text{end} \\ \text{close}: & !\texttt{bye}; \; \text{end} \end{cases}$$

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- Also, Alice's implementation should conform to both $\overline{S_{\mathsf{SA}}}$ and $S_{\mathsf{AB}}$.

# Session Type Duality, Formally

Given a (finite) session type $S$, its dual type $\overline{S}$ is inductively defined as follows:

$$\begin{aligned}
\overline{!U;S} &= ?U;\overline{S} \\
\overline{?U;S} &= !U;\overline{S} \\
\overline{\&\{l_i : S_i\}_{i\in I}} &= \oplus\{l_i : \overline{S_i}\}_{i\in I} \\
\overline{\oplus\{l_i : S_i\}_{i\in I}} &= \&\{l_i : \overline{S_i}\}_{i\in I} \\
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**Notice**:

- For recursive types, duality is defined **coinductively** (the dual of $\mu t.S$ is *not* $\mu t.\overline{S}$)

# Many Classes of Session Types

There are many classes/variants of session types:

- We overviewed **binary** sessions, where types describe two-party protocols. (We used two separate types to handle Alice, Bob, and Seller.)
- With **multiparty** sessions, types describe protocols involving more than two participants. There is a single **global type** and multiple **local types**.

# Many Classes of Session Types (contd.)

There are many classes/variants of session types:

- We saw a syntax of types that describes **regular behaviors**:
  in '$?U; S$' and '$!U; S$' we use ';' to compose an **action** and a **protocol**.
- In **context-free** session types, we can freely compose two protocols: '$S_1; S_2$'.
  This is arguably the most expressive class of protocols (to date).

# Many Classes of Session Types (contd.)

There are many classes/variants of session types:

- Session types can be attached to programming languages with **synchronous** and **asynchronous** communication.
  - Synchronous: A message is received as soon as it is sent
  - Asynchronous: Message reception is not immediate

  These operational differences are reflected in types.
  The real world is asynchronous but synchronous is easier to define.

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  These operational differences are reflected in types.
  The real world is asynchronous but synchronous is easier to define.

- Session types are **paradigm-independent**.
  They have been adapted to functional, object-oriented, and actor-based languages with concurrency.

# Taking Stock

Up to here:

- Correctness for communicating programs
- Sessions as protocol specifications
- A formal syntax for session types
- Example: A two-buyer protocol
- Protocol compatibility via duality for session types

**Coming next**:

- Linear logic, in its intuitionistic variant

# Linear Logic

# Outline

# Intuitionistic Logic (IL)

The grammar of propositions:

$$A, B \quad ::= \quad X \mid A \to B \mid A \times B \mid A + B$$

where:

- $X$ is a propositional **constant**
- $A \to B$ denotes **implication**
- $A \times B$ denotes **conjunction** (alternative notation for $A \wedge B$)
- $A + B$ denotes **disjunction** (alternative notation for $A \vee B$)

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- $A + B$ denotes **disjunction** (alternative notation for $A \vee B$)

Assumptions and judgments:

- $\Gamma, \Delta$ denote **assumptions**: sequences of zero or more propositions.
- A **judgment** $\Gamma \vdash A$ means "from assumptions $\Gamma$ we can conclude $A$".

# Rules for Intuitionistic Logic (IL)

Proving judgments of the form $\Gamma \vdash A$.

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  an introduction rule and an elimination rule (this is called **natural deduction**).

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  an introduction rule and an elimination rule (this is called **natural deduction**).
- Using the rules, we aim to construct **derivations** for judgements.

(Later we will see rules in a **sequent calculus**, rather than in natural deduction.)

# Rules for IL

$$\text{Id}\ \frac{}{A \vdash A} \qquad \text{Exchange}\ \frac{\Gamma, \Delta \vdash A}{\Delta, \Gamma \vdash A} \qquad \text{Contraction}\ \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \qquad \text{Weakening}\ \frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

$$\to\text{-I}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \to\text{-E}\ \frac{\Gamma \vdash A \to B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

$$\times\text{-I}\ \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \times B} \qquad \times\text{-E}\ \frac{\Gamma \vdash A \times B \qquad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}$$

$$+\text{-I}_1\ \frac{\Gamma \vdash A}{\Gamma \vdash A + B} \qquad +\text{-I}_2\ \frac{\Gamma \vdash B}{\Gamma \vdash A + B} \qquad +\text{-E}\ \frac{\Gamma \vdash A + B \qquad \Delta, A \vdash C \qquad \Delta, B \vdash C}{\Gamma, \Delta \vdash C}$$

# Example of a Derivation

A derivation for the judgment

$$A \to B, A \vdash A \times B$$

expressed as a **proof tree** (labels for the rules are omitted):

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{\dfrac{}{A \to B \vdash A \to B} \qquad \dfrac{}{A \vdash A}}{A \to B, A \vdash B}}{A, A \to B, A \vdash A \times B}}{\dfrac{A \to B, A, A \vdash A \times B}{A \to B, A \vdash A \times B}}$$

We see how $A$ is used twice, with the help of contraction and exchange rules.

# Alternative Rules

We have just seen:

$$\times\text{-I}\quad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \times B} \qquad\qquad \times\text{-E}\quad \frac{\Gamma \vdash A \times B \qquad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}$$

We can also have:

$$\times\text{-I'}\quad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \times B} \qquad \times\text{-E}_1\quad \frac{\Gamma \vdash A \times B}{\Gamma \vdash A} \qquad \times\text{-E}_2\quad \frac{\Gamma \vdash A \times B}{\Gamma \vdash B}$$

The rules are **equivalent**: they are interderivable using structural rules.

# More Alternative Rules

We have just seen:

$$\text{Id}\ \frac{}{A \vdash A} \qquad \to\text{-I}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \to\text{-E}\ \frac{\Gamma \vdash A \to B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

We can also have:

$$\text{Id'}\ \frac{}{\Gamma, A, \Delta \vdash A} \qquad \to\text{-I}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \to\text{-E'}\ \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

Here again, structural rules ensure that either group of rules can be used.
We prefer to have an explicit treatment of structural rules.

# Commuting Conversions

The order in which certain (structural) rules are applied is irrelevant.

An equivalence relation on proofs, denoted $\Longleftrightarrow$, via **commuting conversions**.

Here are the commuting conversions involving contraction and $\rightarrow$-E:

$$\dfrac{\dfrac{\Gamma, C, \Delta, C \vdash B}{\Gamma, C, \Delta \vdash B}}{\Gamma, C, \Delta \vdash B} \quad \Longleftrightarrow \quad \dfrac{\dfrac{\Gamma, C, \Delta, C \vdash B}{\Gamma, C, \Delta \vdash B}}{\Gamma, C, \Delta \vdash B}$$

$$\dfrac{\dfrac{\Gamma, \Delta, C, C \vdash B}{\Gamma, \Delta, C \vdash B}}{\Gamma, \Delta, C \vdash B} \quad \Longleftrightarrow \quad \dfrac{\dfrac{\Gamma, \Delta, C, C \vdash B}{\Gamma, \Delta, C \vdash B}}{\Gamma, \Delta, C \vdash B}$$

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$$\dfrac{}{\Gamma, \Delta, C \vdash B} \quad \Longleftrightarrow \quad \dfrac{}{\Gamma, \Delta, C \vdash B}$$

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$$\dfrac{\Gamma \vdash A \to B \qquad \dfrac{\Delta, C, C \vdash A}{\Delta, C \vdash A}}{\Gamma, \Delta, C \vdash B} \quad \Longleftrightarrow \quad \dfrac{\Gamma \vdash A \to B \qquad \Delta, C, C \vdash A}{\dfrac{\Gamma, \Delta, C, C \vdash B}{\Gamma, \Delta, C \vdash B}}$$

# Proof Reduction

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# Proof Reduction

- A judgement may have several, very different derivations.
- In particular, it is not very useful to have an introduction rule for a connective that is followed by the corresponding elimination rule.
- The interest is in **reducing** proofs to avoid unnecessary occurrence of connectives; the proofs themselves may not be "smaller".
- Notions of **normal form** and **subformula property** apply for proofs and their reductions.

# Proof Reduction: Example

Consider the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B} \quad \dfrac{}{B \vdash B}}{B, B \vdash B \times B}}{B \vdash B \times B}}{\vdash B \to (B \times B)} (\dagger) \quad \dfrac{\dfrac{}{A \to B \vdash A \to B} \quad \dfrac{}{A \vdash A}}{A \to B, A \vdash B}}{A \to B, A \vdash B \times B}$$

It can be reduced to:

$$\dfrac{\dfrac{\dfrac{}{A \to B \vdash A \to B} \quad \dfrac{}{A \vdash A}}{A \to B, A \vdash B} \quad \dfrac{\dfrac{}{A \to B \vdash A \to B} \quad \dfrac{}{A \vdash A}}{A \to B, A \vdash B}}{A \to B, A, A \to B, A \vdash B \times B}}{A \to B, A \vdash B \times B} (\dagger)$$

# Propositions as Types

Extending judgements to contain terms (= programs!).

- From the logic perspective, terms encode proofs.
- From the programmer's perspective, terms are a programming language for which logic provides a **type system**.

# Propositions as Types

Propositions can be read as types, and proofs are read as corresponding typed terms:

- A proof of $A \to B$ is a **function** that takes any proof of $A$ into a proof of $B$.
- A proof of $A \times B$ is a **pair** of a proof of $A$ and a proof of $B$.
- A proof of $A + B$ is a **disjoint sum**: either a proof of $A$ or a proof of $B$.

**Terms** encode the rules of the logic:

- An introduction rule: a term that **constructs** a value.
- An elimination rule: a term that **deconstructs** a value.
- There are no term forms for the structural rules.

We briefly have a look at the terms ($\lambda$-calculus!) and their reductions.

# Intuitionistic Terms

$$
\begin{aligned}
s, t, u, v, w \quad ::= \quad & x \\
& \mid \lambda x.\, u \mid s\,(t) \\
& \mid (t, u) \mid \mathtt{case}\; s\; \mathtt{of}\; (x, y) \to v \\
& \mid \mathtt{inl}\,(t) \mid \mathtt{inr}\,(u) \mid \mathtt{case}\; s\; \mathtt{of}\; \mathtt{inl}\,(x) \to v;\; \mathtt{inr}\,(y) \to w
\end{aligned}
$$

Assumptions $\Gamma, \Delta$ are now written in the form

$$x_1 : A_1, \ldots, x_n : A_n$$

where $n \geq 0$ and

- $x_1, \ldots, x_n$ are distinct variables;
- $A_1, \ldots, A_n$ are types (= propositions).

Judgments are then of the form $\Gamma \vdash t : A$ ("term $t$ has type $A$").

# Types for Intuitionistic Terms

Id

$$\frac{}{x : A \vdash x : A}$$

Exchange

$$\frac{\Gamma, \Delta \vdash t : A}{\Delta, \Gamma \vdash t : A}$$

Contraction

$$\frac{\Gamma, y : A, z : A \vdash u : B}{\Gamma, x : A \vdash u[x/y, x/z] : B}$$

Weakening

$$\frac{\Gamma \vdash u : B}{\Gamma, x : A \vdash u : B}$$

$\to$-I

$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x.\, u : A \to B}$$

$\to$-E

$$\frac{\Gamma \vdash s : A \to B \qquad \Delta \vdash t : A}{\Gamma, \Delta \vdash s\,(t) : B}$$

$\times$-I

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash u : B}{\Gamma, \Delta \vdash (t, u) : A \times B}$$

$\times$-E

$$\frac{\Gamma \vdash s : A \times B \qquad \Delta, x : A, y : B \vdash v : C}{\Gamma, \Delta \vdash \mathtt{case}\ s\ \mathtt{of}\ (x, y) \to v : C}$$

# Types for Intuitionistic Terms

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathtt{inl}\,(t) : A + B}\ +\text{-I}_1 \qquad \frac{\Gamma \vdash u : B}{\Gamma \vdash \mathtt{inr}\,(u) : A + B}\ +\text{-I}_2$$

$$\frac{\Gamma \vdash s : A + B \qquad \Delta, x : A \vdash v : C \qquad \Delta, y : B \vdash w : C}{\Gamma, \Delta \vdash \mathtt{case}\ s\ \mathtt{of}\ \mathtt{inl}\,(x) \to v;\ \mathtt{inr}\,(y) \to w : C}\ +\text{-E}$$

# Typing Derivations: Example

Recall the derivation for the judgment $A \to B, A \vdash A \times B$, seen before.
Now with terms:

$$\dfrac{\dfrac{}{y : A \vdash y : A} \qquad \dfrac{\dfrac{}{f : A \to B \vdash f : A \to B} \qquad \dfrac{}{z : A \vdash z : A}}{f : A \to B, z : A \vdash f\,(z) : B}}{\dfrac{y : A, f : A \to B, z : A \vdash (y, f\,(z)) : A \times B}{\dfrac{f : A \to B, y : A, z : A \vdash (y, f\,(z)) : A \times B}{f : A \to B, x : A \vdash (x, f\,(x)) : A \times B}}}$$

# Term Reduction

Since terms encode proofs, proof reductions in logic correspond to term reductions:

$$\lambda x.\, u\,(t) \Longleftrightarrow u[t/x]$$
$$\mathtt{case}\,(t, u)\,\mathtt{of}\,(x, y) \to v \Longleftrightarrow u[t/x, v/y]$$
$$\mathtt{case}\,\mathtt{inl}\,(t)\,\mathtt{of}\;\mathtt{inl}\,(x) \to v;\;\mathtt{inr}\,(y) \to w \Longleftrightarrow v[t/x]$$
$$\mathtt{case}\,\mathtt{inr}\,(u)\,\mathtt{of}\;\mathtt{inl}\,(x) \to v;\;\mathtt{inr}\,(y) \to w \Longleftrightarrow w[u/x]$$

**Notice:**
The notion of proof reduction, associated to properties of proof systems, corresponds precisely to term reduction for a model of computation!

# Outline

# Linear Logic: A Substructural Logic for Computation

- Introduced by Jean-Yves Girard in 1987.
- **Observation**: Contraction arbitrarily duplicates assumptions; weakening freely discards them. Not always sensible for computational resources!

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- Introduced by Jean-Yves Girard in 1987.
- **Observation**: Contraction arbitrarily duplicates assumptions; weakening freely discards them. Not always sensible for computational resources!
- Linear logic advocates a **disciplined** use of resources, by controlling contraction and weakening. Many applications in proof theory and computer science.
- A "resource-aware" logic:
  - $A \vdash B$: given $A$-resources, a way to obtain $B$-resources
  - Proof theoretically: **substructural logic**

# Linear Logic

- We are mostly interested in the intuitionistic variant, dubbed ILL.
  We will also have a look at classical LL (CLL).
- We see rules as a sequent calculus, rather than as natural deduction.
  - In natural deduction: introduction and elimination rules
  - In sequent calculi: left and right rules, plus the cut rule

# Linear Logic: Propositions

'$A$ is true' vs 'I have $A$-resources'

| | |
|---|---|
| $A \to B$ | if $A$ is true, then $B$ is true |
| $A \wedge B$ | both $A$ and $B$ are true |

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| | |
|---|---|
| $A \multimap B$ | if you give me $A$, then I can produce $B$ |
| $A \otimes B$ | I have both $A$ and $B$ |
| $A \,\&\, B$ | I can give you either $A$ or $B$ (you control the choice) |
| $A \oplus B$ | I choose to give you either $A$ or $B$ (I control the choice) |

**Examples**:

- $euro \otimes euro \multimap pizza$
- $dough \otimes (sauce \otimes toppings) \multimap pizza$

# Linear Logic: Linearity

$$A \to A \wedge A \qquad A \not\vdash A \otimes A$$

$$A \wedge B \to A \qquad A \otimes B \not\vdash A$$

# Linear Logic: Linearity

$$A \to A \wedge A \qquad A \not\vdash A \otimes A$$

$$A \wedge B \to A \qquad A \otimes B \not\vdash A$$

$$A \wedge (A \to B) \to B \qquad A \otimes (A \multimap B) \multimap B$$

# Linear Logic: Linearity

$$A \to A \wedge A \qquad A \not\multimap A \otimes A$$

$$A \wedge B \to A \qquad A \otimes B \not\multimap A$$

$$A \wedge (A \to B) \to B \qquad A \otimes (A \multimap B) \multimap B$$

$$A \wedge (A \to B) \to A \wedge B \qquad A \otimes (A \multimap B) \not\multimap A \otimes B$$

Linear Logic is **substructural**, resource-aware.

# Outline

# Proof Theory of $\wedge$

Sequent calculus: $\Gamma \vdash A$

$$\frac{}{A \vdash A} \qquad \wedge\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \qquad \wedge\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\mathrm{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \mathrm{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \qquad \mathrm{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

# Proof Theory of $\wedge$

Assertion $A$ holds under the list of hypotheses $\Gamma$.
Intuitively, $\bigwedge \Gamma \implies A$

Sequent calculus: $\Gamma \vdash A$

$$\frac{}{A \vdash A}$$

$$\wedge\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

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# Proof Theory of $\wedge$

Axiom rule

Sequent calculus: $\Gamma \vdash A$

$$\frac{}{A \vdash A}$$

$$\wedge\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

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# Proof Theory of $\wedge$

Sequent calculus: $\Gamma \vdash A$

Right and left rules for $\wedge$

$$\frac{}{A \vdash A}$$

$$\wedge\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\mathrm{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C}$$

$$\mathrm{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$\mathrm{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

# Proof Theory of $\wedge$

Sequent calculus: $\Gamma \vdash A$

$$\frac{}{A \vdash A} \qquad \wedge\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \qquad \wedge\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\mathrm{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \mathrm{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \qquad \mathrm{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

Structural rules

# Proof Theory of $\wedge$

$\Gamma : \mathit{Frml} \to \mathbb{N}$ is a multiset

Sequent calculus: $\Gamma \vdash A$

$$\frac{}{A \vdash A}$$

$$\wedge\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\mathrm{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C}$$

$$\mathrm{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$\mathrm{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

We can omit the exchange rule by treating $\Gamma$ as a multiset

# Generalized Structural Rules

$$\frac{\Gamma, \Gamma \vdash C}{\Gamma \vdash C} \qquad \frac{\Gamma \vdash C}{\Gamma, \Delta \vdash C}$$

Obtained from repeatedly using the usual rules (and exchange).

# Using Structural Rules

$$\frac{}{A \wedge B \vdash A} \qquad \frac{}{A \vdash A \wedge A}$$

# Using Structural Rules

$$\dfrac{\dfrac{}{A, B \vdash A}}{A \wedge B \vdash A} \qquad \dfrac{}{A \vdash A \wedge A}$$

# Using Structural Rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A} \qquad \dfrac{}{A \vdash A \wedge A}$$

# Using Structural Rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A} \qquad \dfrac{\dfrac{}{A, A \vdash A \wedge A}}{A \vdash A \wedge A}$$

# Using Structural Rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A}$$

$$\dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A, A \vdash A \wedge A}}{A \vdash A \wedge A}$$

# Using Structural Rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A} \qquad \dfrac{A \vdash A \qquad A \vdash A}{A, A \vdash A \wedge A}}{A \vdash A \wedge A}$$

The usage of the contraction and weakening rules is crucial!

# Associativity

$$\frac{}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}$$

# Associativity

$$\dfrac{\dfrac{}{A, B, C \vdash (A \wedge B) \wedge C}}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}$$

# Associativity

$$\dfrac{\dfrac{\dfrac{}{A, B \vdash A \wedge B} \qquad C \vdash C}{A, B, C \vdash (A \wedge B) \wedge C}}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}$$

# Associativity

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B} \qquad C \vdash C}{A, B, C \vdash (A \wedge B) \wedge C}}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}$$

# Alternative Rules for $\wedge$

We can replace $\wedge L$ with the following two rules:

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge\mathsf{L} \qquad \longrightarrow \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge\mathsf{L}_1 \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge\mathsf{L}_2$$

These rules **internalize weakening**.

# Alternative Rules for $\wedge$

We can replace $\wedge R$ with the following rule:

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \longrightarrow \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

# Outline

# From Intuitionistic to Linear Logic

- Main idea: treat $\otimes$ the same as $\wedge$, minus contraction and weakening rules.
- The context $\Gamma$ then externalizes $\otimes$ on the meta-level:

$$\Gamma \vdash A \qquad \longrightarrow \qquad \bigotimes \Gamma \implies A$$

- We will recover contraction and weakening later with the ! **modality**.

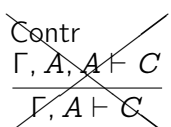
# Proof Theory of $\otimes$

Sequent calculus: $\Gamma \vdash A$

$$\frac{}{A \vdash A}$$

$$\otimes\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

$$\otimes\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$


$$\mathrm{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

# Proof Theory of $\otimes$

Sequent calculus: $\Gamma \vdash A$

Left and right rules for $\otimes$

$$\frac{}{A \vdash A}$$

$$\otimes\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

$$\otimes\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\text{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C}$$

$$\text{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$\text{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

# Proof Theory of $\otimes$

Sequent calculus: $\Gamma \vdash A$

$$\frac{}{A \vdash A} \qquad \otimes\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \qquad \otimes\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\text{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \text{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \qquad \text{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

No weakening or contraction

# Adding the Unit

A proof theory with just $\otimes$ is not fun.
We need multiplicative unit (1) to signify **absence of resources**:

$$\frac{}{\emptyset \vdash 1} \qquad \frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}$$

# Adding Implication

A proof theory with just $\otimes, 1$ is not fun.
We need linear implication ($\multimap$, *lollipop*) to form linear hypotheticals:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$$

# Adding Implication

A proof theory with just $\otimes, 1$ is not fun.
We need linear implication ($\multimap$, *lollipop*) to form linear hypotheticals:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$$

Exercise: try to derive $(A \otimes B) \multimap C \dashv\vdash A \multimap (B \multimap C)$

# Cut rule

$$\frac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\ \text{Cut}$$

What is so special about this rule?
Each cut rule introduces a piece of "complexity" as a new formula $A$

# Cut rule

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Each cut rule introduces a piece of "complexity" as a new formula $A$

## Theorem

*Every proof of* $\Gamma \vdash A$ *that uses cut can be converted into a proof that does not use cut*

# MILL

$$A, B \quad ::= \quad 1 \mid A \otimes B \mid A \multimap B$$

$$\frac{}{A \vdash A} \qquad \frac{}{\emptyset \vdash 1} \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \otimes\mathsf{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes\mathsf{L}$$

$$\frac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\ \mathsf{Cut} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap\mathsf{R} \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}\ \multimap\mathsf{L}$$

# Adding &

Recall the different versions of left/right rules for $\wedge$

$$\wedge\mathrm{R}\quad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

$$\wedge\mathrm{R'}\quad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\wedge\mathrm{L}\quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\wedge\mathrm{L}_1\quad \frac{\Gamma, A_1 \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}$$

$$\wedge\mathrm{L}_2\quad \frac{\Gamma, A_2 \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}$$

# Adding &

Recall the different versions of left/right rules for $\wedge$

$$\otimes\text{R}\quad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

$$\wedge\text{R'}\quad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\otimes\text{L}\quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\wedge\text{L}_1\quad \frac{\Gamma, A_1 \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C} \qquad \wedge\text{L}_2\quad \frac{\Gamma, A_2 \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}$$

# Adding &

Recall the different versions of left/right rules for $\wedge$

$$\otimes\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

$$\&\mathrm{R}\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}$$

$$\otimes\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\&\mathrm{L}_1\ \frac{\Gamma, A_1 \vdash C}{\Gamma, A_1 \,\&\, A_2 \vdash C} \qquad \&\mathrm{L}_2\ \frac{\Gamma, A_2 \vdash C}{\Gamma, A_1 \,\&\, A_2 \vdash C}$$

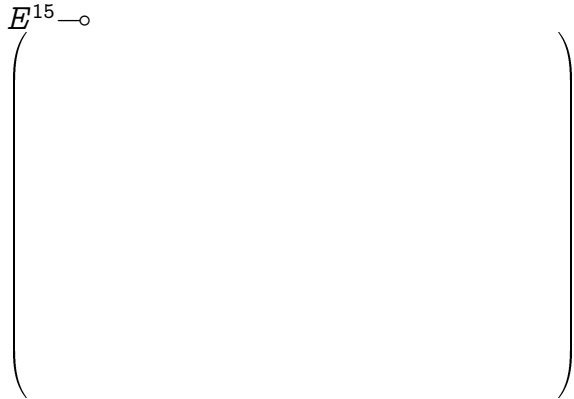
# Interplay of $\otimes$ and $\&$

- $A \,\&\, B \vdash A$, and $A \,\&\, B \vdash B$
- $A \,\&\, B \not\dashv\vdash A \otimes B$
- $(A \multimap B) \,\&\, (A \multimap C) \vdash A \multimap B \,\&\, C$
- $(A \,\&\, B) \multimap C \nvdash A \multimap (B \multimap C)$

# Outline

# Menu at the Linear Logic Cafe

**Summer Menu! €15 p.p.**
**Starter**: Greek Salad, or Soup
**Main**: Chicken Schnitzel, Tofu, or Salmon (for extra €2)
(all main dishes come with a side of fries)
**Desert**: Ice Cream, or Cheese Platter
**Drinks:** Leffe Tripel (for extra €3)



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$$E^{15} \multimap \Big( (Sal \,\&\, Soup) \otimes \Big)$$

**Note:** We use & to indicate that the kitchen can provide any option you choose.

# Menu at the Linear Logic Cafe

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**Starter**: Greek Salad, or Soup
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$$E^{15} \multimap \left( \begin{array}{l} (Sal \,\&\, Soup) \otimes \\ \Big( (Schn \,\&\, Tofu \,\&\, (E \otimes E \multimap Fish)) \end{array} \right)$$

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**Note:** We use & to indicate that the kitchen can provide any option you choose.

# Menu at the Linear Logic Cafe

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$$E^{15} \multimap \left( \begin{array}{l} (Sal \,\&\, Soup) \otimes \\ \Big( (Schn \,\&\, Tofu \,\&\, (E \otimes E \multimap Fish)) \\ \qquad \otimes \; Fries \Big) \otimes \\ (IceCr \,\&\, Cheese) \otimes \\ \\ (E^3 \multimap Beer) \end{array} \right)$$

**Note:** We use & to indicate that the kitchen can provide any option you choose.

# Grandma's Linear Logic Pizza recipe

**LL Pizza Ingredients:**

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

$$\left( \quad \right) \multimap Pizza$$

# Grandma's Linear Logic Pizza recipe

**LL Pizza Ingredients:**

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

$$\left(\begin{array}{l} Yeast \otimes Flour \otimes \\ \\ \\ \\ \\ \end{array}\right) \multimap Pizza$$

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**LL Pizza Ingredients:**

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$$\left(\begin{array}{l} Yeast \otimes Flour \otimes \\ \\ Salt \otimes Sugar \otimes \\ \\ \\ \\ \\ \end{array}\right) \multimap Pizza$$

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- Yeast and flour,
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- Tomatoes
- Sausage or paprika

$$\left(\begin{array}{l} Yeast \otimes Flour \otimes \\ \\ Salt \otimes Sugar \otimes \\ \\ Tomatoes \otimes \\ \\ (Sausage \text{ or } Paprika) \end{array}\right) \multimap Pizza$$

# Grandma's Linear Logic Pizza recipe

**LL Pizza Ingredients:**

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

$$\left(\begin{array}{l} Yeast \otimes Flour \otimes \\ \\ Salt \otimes Sugar \otimes \\ \\ Tomatoes \otimes \\ \\ (Sausage \oplus Paprika) \end{array}\right) \multimap Pizza$$

**Note:** We use $\oplus$ to indicate that the grandma chooses (and we accept her choice).

# Adding $\oplus$

$$\oplus\mathsf{R}_1\ \frac{\Gamma \vdash A_1}{\Gamma \vdash A_1 \oplus A_2} \qquad \oplus\mathsf{R}_2\ \frac{\Gamma \vdash A_2}{\Gamma \vdash A_1 \oplus A_2} \qquad \oplus\mathsf{L}\ \frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 \oplus A_2 \vdash C}$$

# $\oplus$ vs $\&$

The rules are the same as for $\vee$ in the intuitionistic sequent calculus.

# $\oplus$ vs &

The rules are the same as for $\vee$ in the intuitionistic sequent calculus.

However,

$$E \,\&\, (B \oplus C) \nvdash (E \,\&\, B) \oplus (E \,\&\, C)$$

- $E =$ one euro;
- $B =$ beer, $C =$ coffee;
- $(B \oplus C) =$ either beer or coffee, but do not know which;
- $E \,\&\, (B \oplus C) =$ I can either have an euro or a drink.

# $\oplus$ vs $\&$

$$tea \oplus coffee \vdash awake$$

$$tea \,\&\, coffee \vdash awake$$

# $\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

# $\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

# $\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

$tea \,\&\, coffee \vdash tea \oplus coffee$

# $\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

$tea \,\&\, coffee \vdash tea \oplus coffee$

$A \,\&\, B \vdash A \oplus B$ $\qquad$ $A \oplus B \nvdash A \,\&\, B$

# MAILL

$$A, B \quad ::= \quad 1 \mid A \otimes B \mid A \multimap B \mid A \,\&\, B \mid A \oplus B$$

$$\frac{}{A \vdash A} \qquad \frac{}{\emptyset \vdash 1} \qquad \otimes\mathsf{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \qquad \otimes\mathsf{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\mathsf{Cut}\ \frac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B} \qquad \multimap\mathsf{R}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \multimap\mathsf{L}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C} \qquad \&\mathsf{R}\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}$$

$$\&\mathsf{L}_i\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \,\&\, A_2 \vdash C} \qquad \oplus\mathsf{R}_i\ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2} \qquad \oplus\mathsf{L}\ \frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 \oplus A_2 \vdash C}$$

# Related Logics

- Classical Linear Logic

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- Relevance logics
  - Contraction but no weakening
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- Classical Linear Logic
- Relevance logics
  - Contraction but no weakening
  - Intuition: all hypothesis are relevant and must be used
- Logic of Bunched Implications
  - Better known as the kernel of Separation Logic
  - Popular in deductive program verification

# Taking Stock

Up to here:

- Correctness for communicating programs
- Sessions as protocol specifications
- A formal syntax for session types
- Example: A two-buyer protocol
- Protocol compatibility in terms of duality for session types
- **Intuitionistic Linear Logic (MAILL)**

**Tomorrow**:

- Interpreting ILL as session types for message-passing processes